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First natural frequency of multi-segment floor joists with variable cross section*

Key words: floor joist, first natural frequency, Rayleigh method

Introduction

Application of bar elements with variable cross sections provides an opportunity to reduce mass of mechanical systems, thus it is often employed. One of important aspects of the design of building constructions is calculation of the first natural frequency – both of the whole construction and of its separate elements – which is required by law. The papers focused on the determination of forms and frequencies of natural vibrations usually concern elements (beams, bars) having the shape of truncated cone, wedge and multiple-stepped beam. The mode shapes and natural frequencies can be determined from the Euler–Bernoulli differential equation of beam deflection. It can be solved with the Bessel functions of the second kind what, for truncated cone and truncated wedge beams, was presented by Conway and Dubil (1965). Ece, Aydogdu and Taskin (2007) assumed an exponential variability of beam width what enabled to solve the equation of beam vibrations in an exact way, using the method of separation of variables. Naguleswaran (1994) obtained an exact solution for double-tapered beam using the Frobenius method and submitting the tabulated results for various types of beams. For beams of bilinearly varying thickness, Laura, Gutierrez and Rossi (1995) compared values of the dimensionless fundamental frequency obtained by means of three methods: the optimized Rayleigh–Ritz method, the finite element approach and the differential quadrature technique – reaching very good agreement between the Rayleigh–Ritz method and FEM. The multi-segment (stair-shaped) beams were analysed also by Naguleswaran (2002, 2004). Mao (2011) compared the dimensionless frequency from the Naguleswaran’s paper (2002) to

*Due to complexity of the article text was formatted in one-column page style.
his own results obtained by use of the Adomian decomposition method for two-step beam with constant thickness and step-varying width as well as for three step beam with constant width and varying height – reaching the “excellent” agreement. Duan and Wang (2013) studied free vibrations for multiple-stepped beams using the modified discrete singular convolution. Vaz and de Lima Junior (2014) calculated with numerical methods the mode shape of multiple-stepped beams with changes in cross section and compared it with experimental results. Tan, Wang and Jiao (2016) also considered natural (transversal) vibrations of multi-segment beams, obtaining exact results by using transfer matrix method, the exact general solutions of a one-step beam and iterative method.

The first frequency of natural (transversal) vibrations of bars with variable cross section can be estimated using the Rayleigh’s method consisting in comparison of the potential (elastic) and kinetic energy of a vibrating beam. The basic assumption (and obvious simplification) of this method is that the first mode shape is the same as beam deflection due to constant static loads. Making this assumption, Jaworski, Szlachetka and Aguilera-Cortés (2015) analysed cantilever bars with variable cross sections – error of obtained results is small, thus this method can be applied in practice. Such approach was applied to analyse of vibrations of a solid and hollow truncated cone (conical pipe) with generatrices having the shape of straight line and concave parabola (Jaworski & Szlachetka, 2017) as well as convex parabola (Szlachetka, Jaworski & Chalecki, 2017). According to these papers, differences between the results obtained by means of this approach and those obtained with FEM do not exceed 3%.

Using the Rayleigh’s method for beams with variable cross section with assumption that the first mode shape is the same as a shape of the axis of a beam deflection due to constant static loads, one can obtain integrals which often do not have exact solutions or these solutions are described by long equations. In this case, a numerical integration is advantageous – it can be applied for multi-segment bars having shapes described by various functions (cf. e.g. Chalecki, Jaworski, Szlachetka & Bagdasaryan, 2018).

The paper is aimed on presentation of application of such approach in calculations of first natural frequency of multi-segment simply supported beams, symmetrical with respect to their midpoint, having a constant width and variable height. It has been considered a beam consisting generally of five segments (Fig. 1a). Moreover, it has been assumed that the vibration amplitude is small, the material is homogeneous, isotropic and ideally elastic, the bars are slender and the mass is distributed uniformly over the whole bar. The Mathematica software was employed to the calculations. The results were compared to those obtained with FEM software (ANSYS) and for some cases – to the literature data.

Algorithm of proceeding

As the beam depicted in Figure 1a is symmetrical with respect to its midpoint, its natural frequency is the same as for the “halved” beam – sliding in the cut point. Its scheme is shown in Figure 1b.
Cross section area $A(x)$ and second area moment $J(x)$ depend on the beam height $\eta(x)$, thus for consecutive segments are equal to, respectively:

$$
\begin{align*}
A^{(j)}(x) &= b\eta^{(j)}(x), \\
J^{(j)}(x) &= \frac{1}{12}b\left(\eta^{(j)}(x)\right)^3,
\end{align*}
$$

where:
- $b$ – constant width of the cross section;
- $j$ – number of a beam segment ($j = 1, 2, 3$).

Following Jaworski et al. (2015), the beam was loaded by a uniform distributed force. The resulting beam deflection is described by the following second order differential equation of bar elastic deflection curve:

$$
EJ(x)\frac{d^2w(x)}{dx^2} = -M_b(x),
$$

where:
- $E$ – longitudinal modulus of elasticity;
- $w(x)$ – deflection;
- $M_b$ – bending moment in a cross section given by a coordinate $x$.

This equation can be solved with Mathematica software.

The assumption that the neutral axis of a bar deflecting due to the vibrations has a shape described by a function $w(x)$ enables to calculate the potential energy for the
largest deflection and the kinetic energy in the position of equilibrium. The potential energy \( E_p \) and kinetic energy \( E_k \) are equal to, respectively:

\[
E_p = \int_0^L \frac{1}{2} q w(x) dx, \quad E_k = \int_0^L \frac{1}{2} \rho A(x) \omega^2 w^2(x) dx,
\]

where:
- \( q \) – continuous load;
- \( \rho \) – mass density;
- \( \omega \) – natural frequency;

The energy comparison enables to determine the frequency.

The integrands in formula (3) are very complex, thus the integration has been replaced by summation. In this aim, the first, second and third segment of the beam has been divided into \( n_1, n_2 \) and \( n_3 \) elements, respectively (Fig. 2), wherein each element has constant height (equal to a relevant height in the element midpoint) and the same length: \( l_1 = \frac{L_1}{n_1}, \ l_2 = \frac{L_2}{n_2}, \ l_3 = \frac{L_3}{n_3} \), wherein \( L_1, L_2, L_3 \) – lengths of relevant segments.

The number of components of the abovementioned summation (quantities \( n_1, n_2, n_3 \)) must be continuously increased in subsequent iterations. One such iteration encompasses formulas (4)–(14). The summation concerns global deflections of midpoints of each element. Assuming a general shape of the beam shown in Figure 1b, one must execute the following steps to calculate these deflections for a three-segment beam (in the formulas (4)–(13) \( i_1, i_2, i_3 \) – integer numbers, \( i_1 \in \langle 1, n_1 \rangle, \ i_2 \in \langle 1, n_2 \rangle, \ i_3 \in \langle 1, n_3 \rangle \); for a two-segment beam, relevant terms must be omitted).

1. Calculation of reaction forces in the left end of each element

This force is equal to an ordinate of the shear force diagram (Fig. 2b). The variability of shear forces is described by the equation \( T(x) = -qx \). Considering a reaction
for an $i$-th element ($i = i_1, i_2, i_3$), one must discretize this equation by replacing the variable $x$ by the sum of the lengths of all elements from the first one to that having the number $i - 1$, thus

$$R_{i_1}^{(1)} = -qL_1 \frac{i_1 - 1}{n_1}, \quad R_{i_2}^{(2)} = -qL_1 - qL_2 \frac{i_2 - 1}{n_2}, \quad R_{i_3}^{(3)} = -q(L_1 + L_2) - qL_3 \frac{i_3 - 1}{n_3}. \quad (4)$$

2. Calculation of clamp moments in the left end of each element

This moment is equal to an ordinate of the moment diagram (Fig. 2b). The bending moment is described by a quadratic function $M(x) = M_A - \frac{1}{2}qx^2 = \frac{1}{2}qL^2 - \frac{1}{2}qx^2$. Considering a moment for an $i$-th element ($i = i_1, i_2, i_3$), one must discretize this equation by replacing the variable $x$ by the sum of the lengths of all elements from the first one to that having the number $i - 1$, thus

$$M_{i_1}^{(1)} = \frac{1}{2}q(L_1 + L_2 + L_3)^2 - \frac{1}{2}q \left( L_1 \frac{i_1 - 1}{n_1} \right)^2,$$

$$M_{i_2}^{(2)} = \frac{1}{2}q(L_1 + L_2 + L_3)^2 - \frac{1}{2}q \left( L_1 + L_2 \frac{i_2 - 1}{n_2} \right)^2, \quad (5)$$

$$M_{i_3}^{(3)} = \frac{1}{2}q(L_1 + L_2 + L_3)^2 - \frac{1}{2}q \left( L_1 + L_2 + L_3 \frac{i_3 - 1}{n_3} \right)^2.$$

3. Calculation of deflections of right ends of each element in local coordinates

Each element is treated as a cantilever clamped in the left end and loaded by a uniform distributed force $q$ (Fig. 2c). This force evokes a clockwise reaction moment $M_{ij}^{(i)}$ and upward reaction force $R_{ij}^{(i)}$ (its sense corresponds to the signs “-”, appearing in formulas (5)). The deflection of such cantilever is equal to

$$u_{ij}^{(j)} = \frac{1}{EJ^{(j)}} \left( -\frac{1}{2}M_{ij}^{(i)} \left( \frac{L_j}{n_j} \right)^2 - \frac{1}{6}R_{ij}^{(i)} \left( \frac{L_j}{n_j} \right)^3 + \frac{1}{24} q \left( \frac{L_j}{n_j} \right)^4 \right), \quad j = 1, 2, 3. \quad (6)$$

4. Calculation of deflections of midpoints of each element in local coordinates

$$\ddot{u}_{ij}^{(j)} = \frac{1}{EJ^{(j)}} \left( -\frac{1}{2}M_{ij}^{(i)} \left( \frac{L_j}{2n_j} \right)^2 - \frac{1}{6}R_{ij}^{(i)} \left( \frac{L_j}{2n_j} \right)^3 + \frac{1}{24} q \left( \frac{L_j}{2n_j} \right)^4 \right), \quad j = 1, 2, 3. \quad (7)$$
5. Calculation of slopes of right ends of each element in local coordinates

\[ \varphi_{ij}^{(f)} = \frac{1}{EJ^{(f)}} \left( -M_{ij}^{(f)} \frac{L_j}{2n_j} - \frac{1}{2} R_{ij}^{(f)} \left( \frac{L_j}{2n_j} \right)^2 + \frac{1}{6} q \left( \frac{L_j}{2n_j} \right)^3 \right), \quad j = 1, 2, 3. \]  

(8)

6. Calculation of deflections of midpoints of each element in global coordinates

The deflection line of the beam from Figures 1b and 2 has the same shape as a deflection line of a cantilever clamped in the left end and loaded with a continuous force \( q \) on the whole length and an upward concentrated force in the right end, equal to \( q(L_1 + L_2 + L_3) \) – Figure 3, hence at first a way of calculation of deflection \( U \) in global coordinates for that very cantilever will be provided.

\[ U = \frac{1}{n} \sum_{k=1}^{i} \left( l_i \varphi_k + l_i \varphi_{k+1} + \ldots + l_i \varphi_{i+1} \right) + \frac{0.5}{n} \left( 2 \varphi_{i+1} + \varphi_{i+2} + \ldots + \varphi_{i+n} \right) \]  

(9)

**FIGURE 3.** Scheme for derivation of formulas for deflections of midpoints in global coordinates

Figure 4 presents a summation scheme for deflections and slopes needed for the calculation of the midpoint deflection of the element \( i_1 \) of the first segment \( (L_1) \) of the cantilever bar.

\[ U_{i1}^{(f)} = \sum_{k=1}^{i-1} u_k^{(f)} + u_i^{(f)} + \sum_{k=1}^{i-1} \varphi_k^{(f)} (i_1 - k - 0.5) \frac{L_1}{n_1}. \]  

(9)

**FIGURE 4.** Calculation of midpoint deflection of the element \( i_1 \) of the first segment of the cantilever bar

According to such scheme, the midpoint deflection of such element is equal to:
For the second and third segment the expression for the deflection is derived in analogical way but one has to consider additionally the deflection and slope of the end of the previous segment (Fig. 5). For the elements of the second segment, the deflection of the previous segment (the first one) is equal to $U^{(1)}_{n_1} = U^{(1)}_{i_1} \big|_{i=1}^{n_1}$.

Moreover, the deflection resulting from the slope of the midpoint of the last element of the previous segment is equal to:

$$U^{(2)}_{rot1} = \frac{L_1}{2n_1} \sum_{k=1}^{n_1} \varphi_k^{(1)} + \frac{L_2}{n_2} \left( i_2 - \frac{1}{2} \right) \sum_{k=1}^{n_1} \varphi_k^{(1)}.$$ 

Thus, the midpoint deflection of the element $i_2$ of the second segment of the cantilever bar is equal to:

$$U^{(2)}_{i_2} = U^{(1)}_{i_2} \bigg|_{i=1}^{n_1} + \frac{L_1}{2n_1} + \frac{L_2}{n_2} \left( i_2 - \frac{1}{2} \right) \sum_{k=1}^{n_1} \varphi_k^{(1)} + \sum_{k=1}^{i_2-1} u_k^{(2)} + \sum_{k=1}^{i_2-1} \varphi_k^{(2)} (i_2 - k - 0.5) \frac{L_2}{n_2}. ~ \tag{10}$$

In a similar way, the deflections and slopes of the previous segments must be considered for the calculation of the midpoint deflection of the element $i_3$ of the third segment:

$$U^{(3)}_{i_3} = U^{(2)}_{i_2} \bigg|_{i=1}^{n_2} + \frac{L_2}{2n_2} + \frac{L_3}{n_3} \left( i_3 - \frac{1}{2} \right) \left( \sum_{k=1}^{n_1} \varphi_k^{(1)} + \sum_{k=1}^{i_2-1} \varphi_k^{(2)} \right) + \sum_{k=1}^{i_2-1} u_k^{(3)} + \sum_{k=1}^{i_3-1} \varphi_k^{(3)} (i_3 - k - 0.5) \frac{L_3}{n_3}. ~ \tag{11}$$
Now, the appropriate midpoint deflections $w$ of the beam elements can be calculated – according to Figure 3:

$$w_{i1}^{(1)} = U_{i3}^{(3)} \Big|_{i_1 = n_1}, \quad w_{i2}^{(2)} = U_{i3}^{(3)} \Big|_{i_2 = n_2}, \quad w_{ij}^{(j)} = U_{i3}^{(3)} \Big|_{i_j = n_j} - U_{ij}^{(j)},$$

(12)

where $U_{i3}^{(3)} \Big|_{i_1 = n_1} = U_{\text{max}}$.

7. Energies

Having obtained these deflections, one can replace the integration in (3) by a following summation:

$$E_p = \frac{ql}{2} \left[ \sum_{k=1}^{n_1} w_k^{(1)} + \sum_{k=1}^{n_2} w_k^{(2)} + \sum_{k=1}^{n_3} w_k^{(3)} \right],$$

$$E_k = \frac{\rho q^2 l}{2} \left[ \sum_{k=1}^{n_1} A_k^{(1)} \left( w_k^{(1)} \right)^2 + \sum_{k=1}^{n_2} A_k^{(2)} \left( w_k^{(2)} \right)^2 + \sum_{k=1}^{n_3} A_k^{(3)} \left( w_k^{(3)} \right)^2 \right].$$

(13)

8. First natural frequency/period

Comparison of the energies (13) – according to the Rayleigh’s method – yields in the first natural frequency and period as a function of deflection $w$:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{q \rho \left[ \frac{\sum_{k=1}^{n_1} A_k^{(1)} \left( w_k^{(1)} \right)^2 + \sum_{k=1}^{n_2} A_k^{(2)} \left( w_k^{(2)} \right)^2 + \sum_{k=1}^{n_3} A_k^{(3)} \left( w_k^{(3)} \right)^2}{\sum_{k=1}^{n_1} w_k^{(1)} + \sum_{k=1}^{n_2} w_k^{(2)} + \sum_{k=1}^{n_3} w_k^{(3)}} \right]}.$$

(14)

The deflection includes the quantities $q$ and $E$, thus $q$ is being reduced and one obtains the period depending on the parameters $\rho$ and $E$ as well as the beam shape.

This is the end of one iteration. Such iterations must be executed so many times till the relative difference between the results of two last iterations $T$ falls under a certain value chosen by a user (e.g. 0.001%). Due to the accuracy of the results being obtained, it is very advantageous to assume a constant length $l = l_1 = l_2 = l_3$, what means that the quantities $n_1$, $n_2$ and $n_3$ should fulfill a proportion

$$\frac{L_1}{n_1} = \frac{L_2}{n_2} = \frac{L_3}{n_3}.$$ 

Computational examples, accuracy and comparison of results

Two kinds of beams have been investigated: a beam consisting of five rectilinear segments, out of which two have a linearly variable height and the remaining ones – a constant height (cf. Fig. 6a), and a beam consisting of three segments, out of which one has the shape of a parabola convex with respect to the beam axis and the
remaining two have a constant height (cf. Fig. 6b). All beams have constant width. The schemes of such beams contain also the most important geometrical parameters. Such beams are widely applied in building construction – they can play a role of floor joists or spans of pedestrian bridges. In both of these cases, the natural frequency is important because it determines the comfort conditions for people and it is necessary to check a serviceability limit state.

For each beam, its height $\eta(x)$ in individual segments was determined and calculations for the beams from Figure 6 were performed; the results are presented in Tables 1–3.

**TABLE 1. Values of $t$ for the beam from Figure 6a, $L_1 + L_2 = 0.5L$**

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.9H</td>
</tr>
<tr>
<td>0.2($L_1 + L_2$)</td>
<td>8.8573</td>
</tr>
<tr>
<td>0.3($L_1 + L_2$)</td>
<td>8.8623</td>
</tr>
<tr>
<td>0.4($L_1 + L_2$)</td>
<td>8.8731</td>
</tr>
<tr>
<td>0.6($L_1 + L_2$)</td>
<td>8.9086</td>
</tr>
<tr>
<td>0.7($L_1 + L_2$)</td>
<td>8.9322</td>
</tr>
<tr>
<td>0.8($L_1 + L_2$)</td>
<td>8.9536</td>
</tr>
<tr>
<td>0.9($L_1 + L_2$)</td>
<td>8.9803</td>
</tr>
</tbody>
</table>

**TABLE 2. Values of $t$ for the beam from Figure 6a, $L_1 + L_2 = L$**

<table>
<thead>
<tr>
<th>$L_1$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.9H</td>
</tr>
<tr>
<td>0.1($L_1 + L_2$)</td>
<td>9.1467</td>
</tr>
<tr>
<td>0.2($L_1 + L_2$)</td>
<td>9.2109</td>
</tr>
<tr>
<td>0.3($L_1 + L_2$)</td>
<td>9.2579</td>
</tr>
<tr>
<td>0.4($L_1 + L_2$)</td>
<td>9.2921</td>
</tr>
<tr>
<td>0.5($L_1 + L_2$)</td>
<td>9.4119</td>
</tr>
<tr>
<td>0.6($L_1 + L_2$)</td>
<td>9.5102</td>
</tr>
<tr>
<td>0.7($L_1 + L_2$)</td>
<td>9.5872</td>
</tr>
<tr>
<td>0.8($L_1 + L_2$)</td>
<td>9.6898</td>
</tr>
<tr>
<td>$L_1 + L_2$</td>
<td>9.7839</td>
</tr>
</tbody>
</table>

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The vibration period of a beam depicted in Figure 6 can be expressed as:

\[ T = t \frac{L^2}{H} \left( \frac{12}{12} \right)^{1/2} \left( \frac{\rho}{E} \right) \left( \frac{H}{h} \right) \left( \frac{12}{12} \right)^{1/2} \rightarrow t = 8 \frac{12}{H} \left( \frac{h}{H} \right) \frac{12}{12} = \frac{8.8213}{h}. \]

The corresponding values are given in Table 2. The error is below 0.2%.

Naguleswaran (1994) provided the results for beams tapered linearly from one end to another (\( l_1 = 0, l_2 = L \) – special case for the beam from Fig. 6a). Basing on them, one can write:

\[ t = \frac{2 \pi \sqrt{12} L}{k} = 21.7656, \quad \text{where} \quad k \] is a coefficient equal to:

<table>
<thead>
<tr>
<th>( h )</th>
<th>0.9H</th>
<th>0.8H</th>
<th>0.7H</th>
<th>0.6H</th>
<th>0.5H</th>
<th>0.4H</th>
<th>0.3H</th>
<th>0.2H</th>
<th>0.15H</th>
<th>0.1H</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>not provided</td>
<td>2.2196</td>
<td>2.1162</td>
<td>2.0033</td>
<td>1.8744</td>
<td>1.7231</td>
<td>1.5372</td>
<td>1.4230</td>
<td>1.2841</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 3. Values of \( t \) for the beam from Figure 6b**

<table>
<thead>
<tr>
<th>( L )</th>
<th>0.9H</th>
<th>0.8H</th>
<th>0.7H</th>
<th>0.6H</th>
<th>0.5H</th>
<th>0.4H</th>
<th>0.3H</th>
<th>0.2H</th>
<th>0.15H</th>
<th>0.1H</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6H</td>
<td>9.2555</td>
<td>9.9440</td>
<td>10.9056</td>
<td>12.3063</td>
<td>14.5569</td>
<td>18.1910</td>
<td>25.0743</td>
<td>41.3360</td>
<td>60.3471</td>
<td>105.182</td>
</tr>
<tr>
<td>0.8H</td>
<td>9.4835</td>
<td>10.4588</td>
<td>11.8271</td>
<td>13.6672</td>
<td>16.5726</td>
<td>21.1375</td>
<td>29.4877</td>
<td>48.5351</td>
<td>70.2883</td>
<td>120.749</td>
</tr>
</tbody>
</table>

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The values calculated theoretically with use of this coefficient and the corresponding values obtained with use of the procedure presented above are given in two upper grey rows in Table 2. The percentage differences between these values for the given \( h \) vary from 0.19% for \( h = 0.7H \) to 0.66% for \( h = 0.1H \).

To validate obtained results, the authors used FEM (ANSYS) to test the beams shaped as in Figure 6. For the FEM calculations, the plane stress with thickness element was used. It was assumed: length \( L = 10 \) m, height \( H = 0.8 \) m, material – ferroconcrete \((\rho = 2,500 \text{ kg·m}^{-3}, E = 40,000 \text{ MPa})\). According to Figure 6, the periods calculated with FEM are generally slightly longer than those obtained with use of the presented method – differences are lower than 0.7% for the beams from Figure 6a and 1.25% for the beams from Figure 6b.

**FIGURE 6.** Vibration periods for the beams under consideration – comparison of results obtained with the presented method (circles) to those obtained in FEM (crosses) – as well as schemes of those beams along with the most important geometrical parameters.
Conclusions

The paper presents a certain procedure of calculation of first natural frequency of three-segment simply supported beams. It has been shown that for approximated calculations of this frequency the Rayleigh’s method can be applied – with assumption that the shape of the bar axis deflected during vibration is the same as a shape of the axis of a beam deflected by a uniform continuous static load. The accuracy of this procedure is sufficient for engineering calculations. The procedure can be easily extended to multi-segmented beams by addition of appropriate components. As it is apparent from Figure 3, it can be also applied for cantilever bars.

Replacement of a symbolic integration by the presented procedure (summation) allows considerable simplification of calculations. It can be stated that it often enables to carry out calculations because, in many cases, the symbolic integration is unfeasible even for computers. The procedure is quite simple but exact what can be acknowledged its greatest advantage. The differences between the results obtained in FEM and with use of the procedure do not exceed 1.25%, what is an excellent accuracy from an engineering viewpoint.

Acknowledgements

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References


**Summary**

**First natural frequency of multi-segment floor joists with variable cross section.** The Rayleigh’s method can be used to determine the first natural frequency of beams with variable cross-section. The authors analyse multi-segment simply supported beams, symmetrical with respect to their midpoint, having a constant width and variable height. The beams consist generally of five segments. It has been assumed that the neutral bar axis deflected during vibrations has a shape of a beam deflected by a static uniform load. The calculations were made in Mathematica environment and their results are very close to those obtained with FEM.

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